

A single Dirac-fermion Quantum Heat Engine

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We studied the efficiency of two different schemes for a quantum heat engine, by considering a single Dirac fermion trapped in a one-dimensional potential well as the "working substance". The first scheme is a cycle, composed of two adiabatic and two iso-energetic reversible trajectories in configuration space. The trajectories are driven by a quasi-static deformation of the potential well due to an external applied force. The second scheme is a variant of the former, where iso-energetic trajectories are replaced by isothermal ones, along which the system is in contact with macroscopic thermostats. This second scheme constitutes a quantum analogue of the classical Carnot cycle. Our expressions, as obtained from the Dirac single-particle spectrum, converge in the non-relativistic limit to some of the existing results in the literature for the Schrödinger spectrum.

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I. INTRODUCTION

A classical heat engine consists of a cyclic sequence of reversible transformations over a "working substance", typically a macroscopic mass of fluid enclosed in a cylinder with a mobile piston at one end [1, 2]. The two most famous examples are the Otto and Carnot cycles. In particular, the classical Carnot cycle comprises four stages, two isothermal and two adiabatic (iso-entropic) ones. Ideal quasi-static and reversible conditions are achieved by assuming that an external force, which differs only infinitesimally from the force exerted by the internal pressure of the fluid, is applied to the piston in order to let it move extremely slowly [1, 2]. On the other hand, the isothermal trajectories are performed by bringing the fluid contained by the cylinder into thermal equilibrium with external reservoirs at temperatures $T_C < T_H$, respectively.

A quantum analogue of a heat engine involves a sequence of transformations (trajectories) in Hilbert's space, where the "working substance" is of quantum mechanical nature [3–10]. One of the simplest conceptual realizations of this idea is a system composed by a single-particle trapped in a one-dimensional potential well [3–5, 8]. The different trajectories are driven by a quasi-static deformation of the potential well, due to the application of an external force. Two different schemes of this process have been discussed in the literature, in the context of a non-relativistic particle whose energy eigenstates are determined by the Schrödinger spectrum [3, 4, 9]. In this paper, we shall revisit these approaches, and we will study the performance of the corresponding heat engine for a single Dirac fermion. Since Dirac's equation describes the spectra of relativistic particles, results obtained for this case should reduce to the corresponding ones from Schrödinger's equation in the non-relativistic limit. As we shall discuss below, the transition between the relativistic and non-relativistic regimes is determined by the ratio λ/L , with $\lambda = 2\pi\hbar/mc$ the Compton wavelength of the particle, and L the width of

the potential well. The non-relativistic limit corresponds to the regime where $\lambda/L \ll 1$, while evidence of the underlying relativistic nature of the spectrum manifests in terms of finite corrections in powers of λ/L . Another limit of theoretical interest is the "ultra-relativistic" case of massless Dirac fermions, where $\lambda/L \rightarrow \infty$. An important realization of this former case in solid state systems is provided by conduction electrons in the vicinity of the so called Dirac point in graphene [11–14].

II. A DIRAC FERMION TRAPPED IN A ONE-DIMENSIONAL INFINITE POTENTIAL WELL

We define a one-dimensional potential well by the function

$$V(x) = \lim_{V_0 \rightarrow \infty} V_0 [\Theta(x - L) + \Theta(-x)], \quad (1)$$

with $\Theta(x)$ the Heaviside step function. The eigenvalue problem for a Dirac fermion trapped in this potential well is stated as follows

$$\hat{H}\hat{\psi} = E\hat{\psi}, \quad (2)$$

where the Dirac Hamiltonian in 1+1 dimensions is [15]

$$\hat{H} = -i\hbar\hat{\sigma}_1\partial_x + mc^2\hat{\sigma}_3 + V(x)\hat{\mathbf{1}}. \quad (3)$$

Here, $\hat{\sigma}_1$ and $\hat{\sigma}_3$ are Pauli matrices, while $\hat{\psi}(x) = (\phi(x), \chi(x))$ is a two-component spinor.

The solution, considering the boundary condition at the infinite barrier interfaces $\phi(0) = \phi(L) = 0$, is given by

$$\hat{\psi}_n(x) = A \begin{pmatrix} \sin(n\pi x/L) \\ -\frac{i\hbar c(n\pi/L)}{E_n + mc^2} \cos(n\pi x/L) \end{pmatrix}, \quad (4)$$

with associated energy eigenvalues organized in pairs with opposite signs [15],

$$E_n^D(L) = \pm mc^2 \left(\sqrt{1 + (n\lambda/2L)^2} - 1 \right). \quad (5)$$

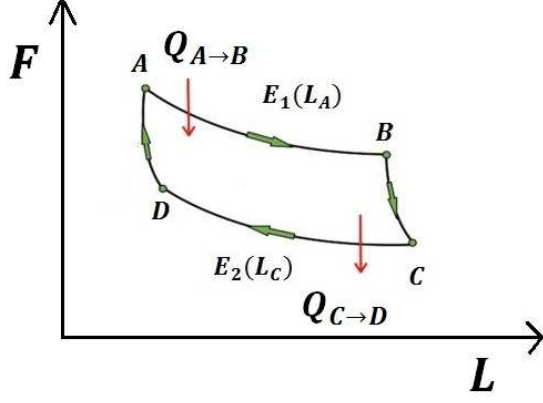


FIG. 1: (Color online) Pictorial description of the Force versus size of the potential well, for the E-Carnot Cycle described in the text.

Here, we have subtracted the rest energy, and $\lambda = 2\pi\hbar/(mc)$ is the Compton wavelength. The positive (negative) sign corresponds to the particle (anti-particle) solution, respectively [15]. In what follows, we shall assume that a single particle is trapped by the potential well, and hence we shall keep the positive eigenvalue.

The spectrum predicted by Eq.(5) can be compared with the corresponding Schrödinger problem, whose eigenvalues are given by

$$E_n^S(L) = \frac{n^2\pi^2\hbar^2}{2mL^2}. \quad (6)$$

Here, a single eigenvalue for the energy is obtained and, moreover, it scales as n^2 , in contrast with the Dirac particle case where a richer scaling with n is observed. In particular, in the regime $\lambda/L \ll 1$, we have:

$$E_n^D(L) \rightarrow \frac{mc^2}{2} (n\lambda/2L)^2 = E_n^S(L), \quad (7)$$

which corresponds to the non-relativistic limit. Beyond this regime, relativistic corrections depending on the finite ratio λ/L are observed. Another interesting limit of Eq.(5) corresponds to a massless Dirac particle with $\lambda \rightarrow \infty$, where the spectrum reduces to the expression

$$E_n^D(L)|_{m=0} = \frac{n\pi\hbar c}{L}. \quad (8)$$

This situation may be of interest in graphene systems, where conduction electrons in the vicinity of the so called Dirac point can be described as effective massless chiral particles, satisfying Dirac's equation in two dimensions[11–14].

III. A SINGLE-PARTICLE QUANTUM HEAT ENGINE

As the "working substance" for a quantum heat engine, let us consider a statistical ensemble of copies of a single-

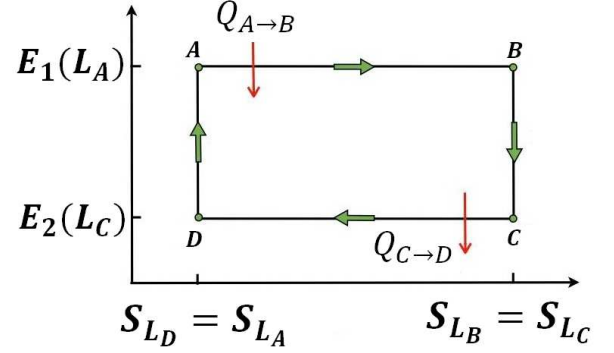


FIG. 2: (Color online) Pictorial description of the Energy versus Entropy (von Neumann) for the E-Carnot cycle.

particle system, where each copy may be in any of the possible different energy eigenstates. We therefore say that the single-particle system is in a statistically mixed quantum state[16]. The corresponding density matrix operator is $\hat{\rho} = \sum_n p_n(L) |\psi_n(L)\rangle \langle \psi_n(L)|$, with $|\psi_n(L)\rangle$ an eigenstate of the single-particle Hamiltonian Eq.(3), corresponding to the spinors defined by Eq.(4). This density matrix operator is stationary, since in the absence of an external perturbation[16] $i\hbar\partial_t\hat{\rho} = [\hat{H}, \hat{\rho}] = 0$. Here, the coefficient $0 \leq p_n(L) \leq 1$ represents the probability for the system, within the statistical ensemble, to be in the particular state $|\psi_n(L)\rangle$. Therefore, the $\{p_n(L)\}$ satisfy the normalization condition

$$\text{Tr}\hat{\rho} = \sum_n p_n(L) = 1. \quad (9)$$

In the context of Quantum Statistical Mechanics, entropy is defined according to von Neumann [16, 17] as $S = -k_B \text{Tr}\hat{\rho} \ln \hat{\rho}$. Since in the energy eigenbasis the density matrix operator is diagonal, the entropy reduces to the explicit expression

$$S(L) = -k_B \sum_n p_n(L) \ln(p_n(L)). \quad (10)$$

In our notation, we emphasize explicitly the dependence of the energy eigenstates $\{|\psi_n(L)\rangle\}$, as well as the probability coefficients $\{p_n(L)\}$, on the width of the potential well L . The ensemble-average energy of the quantum single-particle system is

$$E \equiv \langle \hat{H} \rangle = \text{Tr}(\hat{\rho}\hat{H}) = \sum_n p_n(L) E_n(L). \quad (11)$$

For the statistical ensemble just defined, we conceive two different schemes for a quantum analogue of a thermodynamic heat engine. The first one, that we shall refer to as the E-Carnot cycle, consists on four stages of reversible trajectories: two iso-entropic and two iso-energetic ones. During the iso-energetic trajectories, the ensemble-average energy Eq.(11) is conserved, while

during the iso-entropic ones, the von Neumann entropy defined by Eq.(10) remains constant. We distinguish this first scheme from the quantum Carnot cycle to be discussed next, where the iso-energetic trajectories in Hilbert's space are replaced by isothermal processes. During these stages, the system is brought into thermal equilibrium with macroscopic reservoirs at temperatures $T_C \leq T_H$, respectively.

IV. THE E-CARNOT CYCLE

The system trajectories in Hilbert's space are assumed to be driven by reversible quasi-static processes, in which the walls of the potential well are deformed "very slowly" by an applied external force, such that the distance L is modified accordingly. Along these trajectories, the total change in the ensemble average energy of the system is given by

$$\begin{aligned} dE &= \sum_n p_n(L) dE_n(L) + \sum_n dp_n(L) E_n(L) \\ &= (\delta E)_{\{p_n(L)\}=cnt.} + (\delta E)_{\{E_n(L)\}=cnt.} \end{aligned} \quad (12)$$

The first term in Eq.(12) represents the total energy change due to an iso-entropic process, whereas the second term represents a trajectory where the energy spectrum remains rigid.

Let us consider first the iso-entropic process, where $\{p_n(L)\} = cnt.$. We remark that this represents a strong sufficient condition for the entropy to remain constant along the trajectory, but is not a necessary one[18]. Under quasi-static conditions, the external force driving the change in the width of the potential well is equal to the internal "pressure" of the one-dimensional system, $F = -(\partial E / \partial L)_S$. Therefore, the work performed by the system against the external force, when the width of the potential well expands from $L = L_1$ to $L = L_2$, is given by

$$\begin{aligned} W_{1 \rightarrow 2} &= \int_{L_1}^{L_2} dL \left(\frac{\partial E}{\partial L} \right)_{\{p_n(L)\}=\{p_n(L_1)\}=cnt.} \\ &= \sum_{n=1}^{\infty} p_n(L_1) [E_n(L_2) - E_n(L_1)]. \end{aligned} \quad (13)$$

For the case of a Dirac fermion, the work performed under iso-entropic conditions is given after Eq.(13) and Eq.(5) by

$$\begin{aligned} W_{1 \rightarrow 2} &= mc^2 \sum_{n=1}^{\infty} p_n(L_1) \left(\sqrt{1 + (n\lambda/2L_2)^2} \right. \\ &\quad \left. - \sqrt{1 + (n\lambda/2L_1)^2} \right). \end{aligned} \quad (14)$$

Notice that our sign convention is such that, for an expansion process $L_2 > L_1$, the work performed by the system is negative[2], indicating that the ensemble-averaged

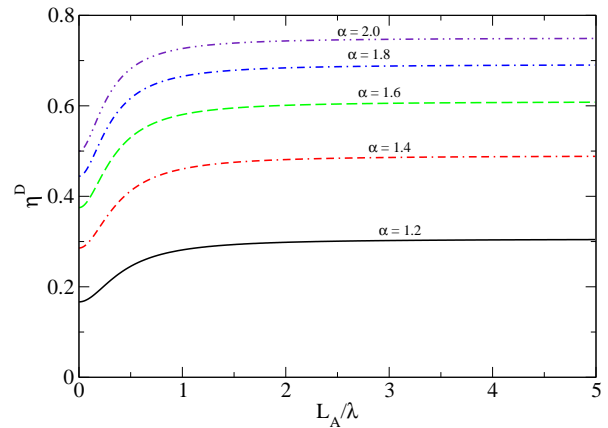


FIG. 3: (Color online) Efficiency of the E-Carnot cycle, calculated after Eq.(28), as a function of the expansion parameter $\alpha = L_C/L_B$.

energy is decreasing during expansion, as in a classical ideal gas.

Let us now consider an iso-energetic process, that is, a trajectory in Hilbert space defined by the equation $dE = 0$. The solution to this equation, for $L \in [L_1, L_2]$, is given by the path

$$\sum_{n=1}^{\infty} p_n(L) E_n(L) = \sum_{n=1}^{\infty} p_n(L_1) E_n(L_1), \quad (15)$$

along with the normalization condition Eq.(9). Clearly, by definition an iso-energetic process satisfies

$$dE = \delta W_{1 \rightarrow 2} + \delta Q_{1 \rightarrow 2} = 0, \quad (16)$$

with $\delta W_{1 \rightarrow 2} \equiv (\delta E)_{\{p_n(L)\}=cnt.}$ and $\delta Q_{1 \rightarrow 2} \equiv (\delta E)_{\{E_n(L)\}=cnt.}$. In the former equation, the integral along the trajectory $L_1 \rightarrow L_2$ gives

$$\Delta E = W_{1 \rightarrow 2} + Q_{1 \rightarrow 2} = 0. \quad (17)$$

The first term $W_{1 \rightarrow 2}$ corresponds to the mechanical work performed by the system against the external force, which drives the change in the width of the potential well at constant energy. The second term $Q_{1 \rightarrow 2} = -W_{1 \rightarrow 2}$ corresponds to the amount of energy exchanged by the system with the environment, in order to rearrange its internal level occupation. This equation is in analogy with the first law of thermodynamics for macroscopic systems, when considering a reversible process over a classical ideal gas which is being compressed/expanded at constant internal energy conditions. The first term has a precise correspondence with the mechanical work for expansion/compression, whereas the second is in correspondence with the heat exchanged by the gas with the environment in order to satisfy total energy conservation.

According to the previous analysis, the heat exchanged by the system with the environment along the iso-

energetic process is given by

$$Q_{1 \rightarrow 2} = \sum_{n=1}^{\infty} \int_{L_1}^{L_2} E_n(L) \frac{dp_n(L)}{dL} dL. \quad (18)$$

Evidently, Eq.(15) combined with the normalization condition Eq.(9) are not enough to uniquely define the coefficients $p_n(L)$ along the iso-energetic trajectory. An exception is the case when the energy scale of all the processes involved is such that only transitions between the ground state ($n = 1$) and the first excited state ($n = 2$) are possible. In this effective two-level spectrum, combining Eq.(15) with the normalization condition Eq.(9), the trajectory for the iso-energetic process is described by the following relation

$$p_1(L) = \frac{E_2(L_1) - E_2(L)}{E_1(L) - E_2(L)} + \frac{E_1(L_1) - E_2(L_1)}{E_1(L) - E_2(L)} p_1(L_1), \quad (19)$$

with $p_2(L) = 1 - p_1(L)$ after the normalization condition Eq.(9). The heat exchanged by the system with the environment during an iso-energetic trajectory connecting the initial and final states $L_1 \rightarrow L_2$, for the case of a Dirac particle, is given by the expression

$$\begin{aligned} -Q_{1 \rightarrow 2} = & [E_2^D(L_1) + (E_1^D(L_1) - E_2^D(L_1)) \\ & \times p_1(L_1)] \ln \left[\frac{E_1^D(L_2) - E_2^D(L_2)}{E_1^D(L_1) - E_2^D(L_1)} \right] + 2(mc^2) \ln \left(\frac{L_2}{L_1} \right) \\ & + (mc^2) \ln \left[\frac{mc^2 + E_2^D(L_2)}{mc^2 + E_2^D(L_1)} \frac{mc^2 + E_1^D(L_2)}{mc^2 + E_1^D(L_1)} \right] \end{aligned} \quad (20)$$

where $E_n^D(L)$ was defined in Eq.(5).

For the effective two-level system previously described, we shall conceive a cycle, as depicted in Figure 1, which starts in the ground state with $p_1(L_A) = 1$. Then, the system experiences an iso-energetic expansion from $L_A \rightarrow L_B > L_A$. Then, it experiences an iso-entropic expansion from $L_B \rightarrow L_C > L_B$, then an iso-energetic compression $L_C \rightarrow L_D < L_C$, and finally it goes back to the initial ground state through an iso-entropic compression $L_D \rightarrow L_A$.

We shall assume that the final state after the iso-energetic process $L_A \rightarrow L_B$ corresponds to maximal expansion, that is, the system ends completely localized in the excited state $n = 2$. In this condition, Eq.(19) reduces to

$$p_1(L_B) = 0, \quad p_2(L_B) = 1. \quad (21)$$

The condition of total energy conservation between the initial and final states connected through an iso-energetic process, for maximal expansion, leads to the equation

$$p_1(L_A)E_1(L_A) = p_2(L_B)E_2(L_B), \quad (22)$$

where $p_1(L_A) = p_2(L_B) = 1$ for maximal expansion. Therefore Eq.(22), given the Dirac spectrum Eq.(5), implies that $L_B/L_A = 2$.

The heat released to the environment along this first stage of the cycle is calculated after Eq.(20),

$$\begin{aligned} -Q_{A \rightarrow B} = & E_1^D(L_A) \ln \left[\frac{E_2^D(L_A) - E_1^D(L_A)}{E_2^D(2L_A) - E_1^D(2L_A)} \right] \\ & + mc^2 \ln \left[\frac{1}{4} \left\{ \frac{mc^2 + E_2^D(L_A)}{mc^2 + E_1^D(2L_A)} \right\} \right]. \end{aligned} \quad (23)$$

The next process is an iso-entropic expansion, characterized by the condition $p_2(L_B) = p_2(L_C) = 1$. We shall define the expansion parameter $\alpha \equiv L_C/L_B > 1$. The work performed during this stage, with $L_B = 2L_A$ (as discussed before), is calculated from Eq.(14),

$$W_{B \rightarrow C} = mc^2 \left[\sqrt{1 + \left(\frac{\lambda}{2L_A} \right)^2} - \sqrt{1 + \left(\frac{\lambda}{2\alpha L_A} \right)^2} \right]. \quad (24)$$

The cycle continues with a maximal compression process from $L_C = 2\alpha L_A$ to $L_D = \alpha L_A$ under iso-energetic conditions. The energy conservation condition is in this case similar to Eq.(22), with $p_2(L_C) = p_1(L_D) = 1$. The heat exchanged by the system with the environment along this process, applying Eq.(20), is given by the expression

$$\begin{aligned} -Q_{C \rightarrow D} = & E_2^D(2\alpha L_A) \ln \left[\frac{E_2^D(\alpha L_A) - E_1^D(\alpha L_A)}{E_2^D(2\alpha L_A) - E_1^D(2\alpha L_A)} \right] \\ & + mc^2 \ln \left[\frac{1}{4} \left\{ \frac{mc^2 + E_2^D(\alpha L_A)}{mc^2 + E_1^D(2\alpha L_A)} \right\} \right] \end{aligned} \quad (25)$$

where $E_n^D(L)$ was defined in Eq.(5). The last path along the cycle is an adiabatic process, which returns the system to its initial ground state with $p_1(L_D) = p_1(L_A) = 1$. The work performed during this final stage, as obtained by applying Eq.(14), is given by

$$W_{D \rightarrow A} = mc^2 \left[\sqrt{1 + \left(\frac{\lambda}{2\alpha L_A} \right)^2} - \sqrt{1 + \left(\frac{\lambda}{2L_A} \right)^2} \right]. \quad (26)$$

It is interesting to check that the work along the two iso-entropic trajectories cancels, that is $W_{B \rightarrow C} + W_{D \rightarrow A} = 0$. Therefore, the efficiency of the cycle is defined by the ratio

$$\eta^D = 1 - \frac{Q_{C \rightarrow D}}{Q_{A \rightarrow B}}. \quad (27)$$

When substituting the corresponding expressions from Eq.(23) and Eq.(25) into Eq.(27), we obtain the explicit analytical expression

$$\eta^D = 1 - \frac{\ln \left[\frac{1}{4} \frac{1 + \theta(\alpha L_A/2)}{1 + \theta(2\alpha L_A)} \left\{ \frac{\theta(\alpha L_A/2) - \theta(\alpha L_A)}{\theta(\alpha L_A) - \theta(2\alpha L_A)} \right\}^{\theta(\alpha L_A)} \right]}{\ln \left[\frac{1}{4} \frac{1 + \theta(L_A/2)}{1 + \theta(2L_A)} \left\{ \frac{\theta(L_A/2) - \theta(L_A)}{\theta(L_A) - \theta(2L_A)} \right\}^{\theta(L_A)} \right]}. \quad (28)$$

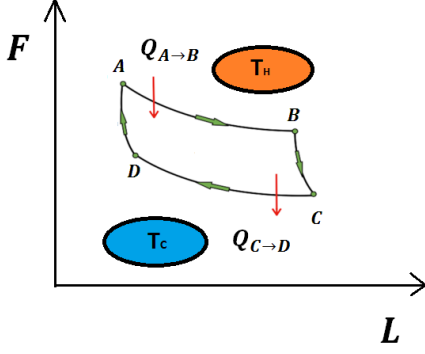


FIG. 4: (Color online) Pictorial description of the Force versus width of the potential well, for the Quantum Carnot Cycle described in the text.

Here, we have defined the function $\theta(L) = \sqrt{1 + (\lambda/2L)^2}$, with $\lambda = 2\pi\hbar/(mc)$ the Compton wavelength. It is important to remark that in the non-relativistic limit $\lambda/L \rightarrow 0$, the expression in Eq.(27) reduces to the known Schrödinger limit

$$\lim_{\lambda/L \rightarrow 0} \eta^D = 1 - 1/\alpha^2. \quad (29)$$

In the "ultra-relativistic" case of a massless Dirac fermion, $\lambda/L \rightarrow \infty$, the efficiency converges to the length independent limit

$$\lim_{\lambda/L \rightarrow \infty} \eta^D = 1 - 1/\alpha. \quad (30)$$

The trend of the efficiency is shown in between both limits in Figure 3. It is worth to remark that, since the expansion parameter $\alpha = L_C/L_B > 1$, the efficiency in the strict non-relativistic (Schrödinger) regime $\lambda/L \rightarrow 0$ Eq.(29) is the highest possible one. This is also clear from the asymptotics of the curves displayed in Fig. 3, where the "ultra-relativistic" limit corresponding to massless fermions ($\lambda/L \rightarrow \infty$) indeed represents the less efficient regime for a fixed expansion parameter α .

V. THE QUANTUM CARNOT CYCLE

In this section, we shall discuss the quantum version of the Carnot cycle, as applied to the statistical ensemble of Dirac single-fermion systems under consideration. The thermodynamic cycle which defines the corresponding heat engine is composed of four stages or trajectories in Hilbert's space: Two isothermal and two iso-entropic processes.

In the first stage, the system is brought into contact with a thermal reservoir at temperature T_H . By keeping isothermal conditions, the width of the potential well is expanded from $L_A \rightarrow L_B$. Since thermal equilibrium with the reservoir is assumed along this process, the von

Neumann entropy of the system achieves a maximum for the Boltzmann distribution [16, 17]

$$p_n(L, \beta_H) = [Z(L, \beta_H)]^{-1} e^{-\beta_H E_n^D(L)}, \quad (31)$$

with $\beta = (k_B T)^{-1}$, and the normalization factor is given by the partition function

$$Z(L, \beta) = \sum_{n=0}^{\infty} e^{-\beta E_n^D(L)} \sim \frac{2L}{\lambda} e^{\beta m c^2} K_1(\beta m c^2). \quad (32)$$

Here, the second expression, as shown in Appendix, is the continuum approximation to the discrete sum, valid in the physically relevant regime $\lambda \ll L$. Here, $K_1(x)$ is a Bessel function of the second kind.

From a similar analysis as in the previous section, we conclude that the heat exchanged by the system to the thermal reservoir is given by

$$\begin{aligned} Q_{A \rightarrow B} &= \int_{L_A}^{L_B} \sum_{n=0}^{\infty} E_n(L) \frac{dp_n(L, \beta_H)}{dL} dL \\ &= - \frac{\partial \ln \left(\frac{Z(L_B, \beta_H)}{Z(L_A, \beta_H)} \right)}{\partial \beta_H} + \beta_H^{-1} \ln \left(\frac{Z(L_B, \beta_H)}{Z(L_A, \beta_H)} \right) \\ &= m c^2 \ln \left(\frac{L_B}{L_A} \right). \end{aligned} \quad (33)$$

In the second line, we have done integration by parts, and we made direct use of the definition Eq.(30) of the partition function. The final result follows from substituting the explicit expression for the partition function Eq.(32).

Similarly, during the third stage of the cycle, the system is again brought into contact with a thermal reservoir, but at a lower temperature $T_C < T_H$. Therefore, the probability distribution of states in the ensemble is $p_n(L, \beta_C)$, as defined in Eq.(31), but with T_C instead of T_H . The heat released to the reservoir during this stage is given by the expression

$$Q_{C \rightarrow D} = m c^2 \ln \left(\frac{L_D}{L_C} \right). \quad (34)$$

The second and fourth stages of the cycle constitute iso-entropic trajectories. In order to analyze these stages, we shall derive the "equation of state" for the statistical ensemble of single-particle systems. When substituting the Boltzmann distribution $p_n(\beta, L) = [Z(\beta, L)]^{-1} \exp(-\beta E_n^D(L))$ into the expression for the von Neumann entropy Eq.(10), we obtain the relation

$$S = T^{-1} E + k_B \ln Z(\beta, L). \quad (35)$$

Here, $E = \langle \hat{H} \rangle$ is the ensemble-average energy, as defined by Eq.(11). The equation of state is obtained from Eq.(35) as

$$\begin{aligned} F &= - \left(\frac{\partial E}{\partial L} \right)_S = k_B T \frac{\partial}{\partial L} \ln Z(\beta, L) \\ &= \frac{k_B T}{L}. \end{aligned} \quad (36)$$

In the last line, we have used the explicit analytical expression Eq.(32) for the partition function to calculate the derivative. The equation of state Eq.(36) reflects that the ensemble of systems behaves as a one-dimensional ideal gas. This is not surprising, since the ensemble-average energy is given by

$$\begin{aligned} E &= \langle \hat{H} \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta, L) \\ &= mc^2 \left(-1 - \frac{d}{dz} \ln K_1(z) \right), \end{aligned} \quad (37)$$

where we have substituted explicitly the expression for the partition function, and in the final step we defined $z = \beta mc^2$. Here, $K_1(x)$ is a Bessel function of the second kind. Eq.(37) shows that the ensemble average energy of the system is a function of the temperature solely, from which the ideal gas equation of state follows as a natural consequence. We can thus define the "specific heat" at constant length, which after Eq.(37) is given by

$$C_L = \left(\frac{\partial E}{\partial T} \right)_L = \frac{dE}{dT} = k_B z^2 \frac{d^2}{dz^2} \ln K_1(z), \quad (38)$$

where $z = \beta mc^2$. It is interesting to remark that, based on the asymptotic behaviour of the Bessel functions $K_n(z) \sim \sqrt{\pi/2} z^{-1/2} \exp(-z) + O(z^{-1})$, the specific heat defined in Eq.(38) presents the asymptotic limit $C_L \rightarrow k_B/2$ when $k_B T \ll mc^2$. This is the well known result for a classic non-relativistic ideal gas in one dimension. This feature and the general temperature dependence of the ensemble specific heat is displayed in Fig. 6. The change in the ensemble averaged energy of the system, for a general process, is $dE = TdS - FdL$. Since the ensemble-average energy is a function of temperature only, the differential equation for an iso-entropic trajectory ($dS = 0$) is [19]

$$dE = C_L dT = -FdL = -k_B T \frac{dL}{L}. \quad (39)$$

Separating variables, after some algebra we obtain

$$z \frac{d^2}{dz^2} \ln K_1(z) = \frac{dL}{L}. \quad (40)$$

Integrating Eq.(40) between initial conditions (z_0, L_0) and final conditions (z, L) , we have

$$\frac{L}{L_0} = \frac{K_1(z_0)}{K_1(z)} e^{z \frac{K_0(z)}{K_1(z)} - z_0 \frac{K_0(z_0)}{K_1(z_0)}}. \quad (41)$$

Here, we made use of the Bessel function identity $K_1'(z) = K_0(z) - z^{-1}K_1(z)$, with $z = \beta mc^2$. It is interesting to check that in the non-relativistic limit $z \gg 1$, given the asymptotic behaviour of the Bessel functions $K_n(z) \sim \sqrt{\pi/2} z^{-1/2} \exp(-z) + O(z^{-1})$, Eq.(41) reduces to $LT^{-1/2} = \text{const.}$ for the iso-entropic trajectory. On the other hand, in the "ultra-relativistic" limit of a massless

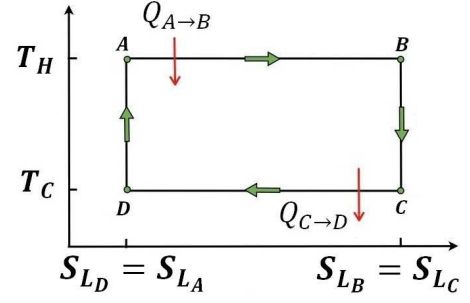


FIG. 5: (Color online) Pictorial description of the Force versus width of the potential well for the Quantum Carnot Cycle.

Dirac particle with $z \rightarrow 0$, the iso-entropic trajectory is given by the equation $LT^{-1} = \text{const.}$

We are now in conditions to discuss the second and fourth stages of the Carnot cycle. The second stage of the process corresponds to an iso-entropic trajectory, parameterized in differential and integral form by Eqs.(40) and (41), respectively. The work performed by the system during this process is given by

$$\begin{aligned} W_{B \rightarrow C} &= - \int_{L_B}^{L_C} F dL = - \int_{L_B}^{L_C} k_B T \frac{dL}{L} \\ &= -mc^2 \int_{z_H}^{z_C} \frac{d^2}{dz^2} \ln K_1(z) dz \end{aligned} \quad (42)$$

Here, in the second line we have used the differential equation defining the iso-entropic trajectory, Eq.(40). Evaluating the integral in Eq.(42), we explicitly obtain

$$W_{B \rightarrow C} = k_B (T_C - T_H) - mc^2 \left[\frac{K_0(z_C)}{K_1(z_C)} - \frac{K_0(z_H)}{K_1(z_H)} \right] \quad (43)$$

The fourth and final stage of the cycle also corresponds to an iso-entropic trajectory $L_D \rightarrow L_A$, and the work performed by the system against the external applied force is obtained similarly as in Eq.(42),

$$\begin{aligned} W_{D \rightarrow A} &= - \int_{L_D}^{L_A} F dL \\ &= k_B (T_H - T_C) - mc^2 \left[\frac{K_0(z_H)}{K_1(z_H)} - \frac{K_0(z_C)}{K_1(z_C)} \right]. \end{aligned} \quad (44)$$

Clearly, after Eqs.(43) and (45), we have $W_{B \rightarrow C} + W_{D \rightarrow A} = 0$, and hence the contribution of the work along the iso-entropic trajectories vanishes.

From the equation for the iso-entropic trajectory Eq.(41), we conclude that the length ratios are determined by the temperatures of the thermal reservoirs,

$$\frac{L_C}{L_B} = \frac{L_D}{L_A} = \frac{K_1(z_H)}{K_1(z_C)} e^{z_C \frac{K_0(z_C)}{K_1(z_C)} - z_H \frac{K_0(z_H)}{K_1(z_H)}}. \quad (45)$$

From Eq.(45), we also obtain $L_A/L_B = L_D/L_C$. Substituting this relation in the expression for the efficiency

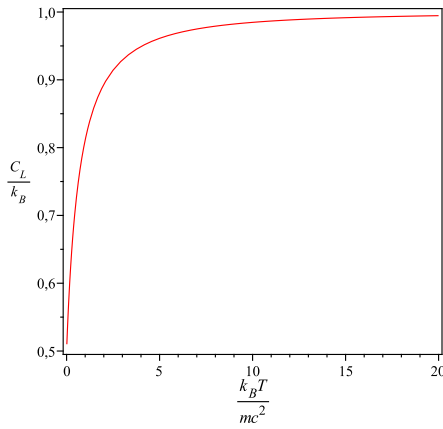


FIG. 6: (Color online) Specific heat of the statistical ensemble of single Dirac-fermion systems, after Eq.(38), as a function of temperature.

of the cycle, from Eq.(33) and Eq.(34) we have

$$\begin{aligned} \eta^C &= 1 - \frac{Q_{C \rightarrow D}}{Q_{A \rightarrow B}} = 1 - \frac{T_C \ln(L_D/L_C)}{T_H \ln(L_B/L_A)} \\ &= 1 - \frac{T_C}{T_H}. \end{aligned} \quad (46)$$

Therefore, we have recovered the expression for the efficiency identical to the classical Carnot cycle.

VI. CONCLUSIONS

By considering as a "working substance" the statistical ensemble for a Dirac single-fermion system trapped in a one-dimensional potential well, we have analyzed two different schemes for a quantum heat engine. The first, that we have referred to as the E-Carnot cycle, consists of two iso-entropic and two iso-energetic trajectories. We obtained an explicit expression for the efficiency of this cycle and showed that our analytical result, in the non-relativistic limit $\lambda/L \rightarrow 0$, reduces to the corresponding one for a Schrödinger particle, as reported in the literature [3]. Our results also indicate that the efficiency for this E-Carnot cycle is higher in the non-relativistic region of parameters, that is for $L \gg \lambda$, when comparing at the same compressibility ratios $\alpha = L_C/L_B > 1$. An exception is the case of massless Dirac fermions, with $\lambda = \infty$, where it is not possible to achieve non-relativistic conditions. This is of potential practical interest for graphene systems, where conduction electrons are indeed described as massless chiral Dirac fermions[11–14].

As a second candidate for a quantum heat engine, we discussed a version of the Carnot cycle, composed by two iso-thermal and two iso-entropic trajectories. In order to achieve iso-thermal conditions, we consider that the single-fermion system is in thermal equilibrium with macroscopic reservoirs at temperatures $T_C < T_H$, respec-

tively. Therefore, the statistical ensemble under consideration is described by the density matrix $\hat{\rho} = e^{-\beta \hat{H}}/Z$, with $Z[\beta, L] = \text{Tr} e^{-\beta \hat{H}}$ the partition function. We showed that the statistical properties of the ensemble are such that an equation of state can be defined, as well as a specific heat, in analogy with a classical ideal gas in one dimension. We obtained the equation for the iso-entropic trajectory, which in the non-relativistic limit $k_B T \ll mc^2$ reduces to the classical result $LT^{-1/2} = \text{cnt}$. On the other hand, we also showed that in the "ultra-relativistic" limit of a massless Dirac fermion, as for instance conduction electrons in graphene, the iso-entropic trajectory is defined by the equation $LT^{-1} = \text{cnt}$. We also showed that the efficiency for the Quantum Carnot cycle satisfies the same relation that the classical one in terms of the temperatures of the thermostats, that is $\eta = 1 - T_C/T_H$.

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Appendix

We here discuss the continuum approximation to the discrete partition function Eq.(30).

$$Z(L, \beta) = \sum_{n=0}^{\infty} e^{-\beta mc^2 \left(\sqrt{1 + \left(\frac{n\lambda}{2L} \right)^2} - 1 \right)}$$

Here, let us define the discrete variable $x_n = n\lambda/(2L)$. The spacing between two consecutive values is given by $\Delta x = x_{n+1} - x_n = \lambda/(2L)$, and hence the expression for the partition function can be written as

$$Z(L, \beta) = \frac{2L}{\lambda} \sum_{n=0}^{\infty} \Delta x e^{-\beta mc^2 \left(\sqrt{1 + x_n^2} - 1 \right)}$$

For physically relevant sizes of the potential well, we shall have $\lambda/L \ll 1$, which allow us to take the continuum limit in the sense of a Riemann sum for the previous equation,

$$\begin{aligned} Z(L, \beta) &\rightarrow \frac{2L}{\lambda} \int_0^{\infty} dx e^{-\beta mc^2 (\sqrt{1+x^2}-1)} \\ &= \frac{2L}{\lambda} \int_0^{\infty} dt e^{-\beta mc^2 (\cosh(t)-1)} \cosh(t) \\ &= \frac{2L}{\lambda} e^{\beta mc^2} K_1(\beta mc^2). \end{aligned}$$

Here, in the second line we have made the substitution $x = \sinh(t)$, and $K_1(x)$ is a Bessel function of the second kind.

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